Multipole field representations

The general formula for multipole potential is

$$\varphi_n = \frac{B_n}{(n!)a^{(n-1)}} \cdot r^n \sin(n \cdot \theta), \tag{1a}$$

where $n = 1$ is for dipole, $n = 2$ for quadrupole, and so on. The skew multipole potential can be written as:

$$\begin{aligned}\psi_n &= \frac{A_n}{(n!)a^{(n-1)}} \cdot r^n \sin\left[n \cdot \left(\theta - \frac{\pi}{2 \cdot n}\right)\right] \\ &= \frac{-A_n}{(n!)a^{(n-1)}} \cdot r^n \cos(n \cdot \theta)\end{aligned} \tag{1b}$$

The use of $n!$ in the denominator ensures that this representation is consistent with multi-pole field definition use in MAD program. The expressions $\frac{B_n}{a^{(n-1)}}$ and $\frac{A_n}{a^{(n-1)}}$ should be understood as the strength of corresponding multipole field. The B_x, B_y, A_x, A_y used in equation (2) and later are to be interpreted as fields measured at a coordinate (x, y).

Dipole

The dipole potential in Cartesian coordinate is quite simple, following equation (1a) with $n = 1$:

$$\varphi_1 = \frac{B_1}{1 \cdot a^0} \cdot r \sin(\theta) = B_1 \cdot y$$

The resulting magnetic fields are exactly that of a horizontal dipole:

$$\begin{aligned}B_x &= \frac{\partial \varphi_1}{\partial x} = 0 \\ B_y &= \frac{\partial \varphi_1}{\partial y} = B_1\end{aligned} \tag{2a}$$

Skew dipole

This is just a fancier name for vertical bending dipole. Its potential can be written down according to equation (1b) as:

$$\psi_1 = A_1 \cdot r \sin\left(\theta - \frac{\pi}{2}\right) = -A_1 \cdot r \cos(\theta) = -A_1 \cdot x$$

And the magnetic fields are:

$$\begin{aligned}A_x &= \frac{\partial \varphi_1}{\partial x} = -A_1 \\ A_y &= \frac{\partial \varphi_1}{\partial y} = 0\end{aligned} \tag{2b}$$

Quadrupole

The quadrupole potential in Cartesian coordinate can be written as:

$$\varphi_2 = \frac{B_2}{2 \cdot a} \cdot r^2 \sin(2 \cdot \theta) = \frac{B_2}{a} \cdot xy$$

From this we derive the magnetic field:

$$\begin{aligned} B_x &= \frac{B_2}{a} \cdot y \\ B_y &= \frac{B_2}{a} \cdot x \end{aligned} \tag{3a}$$

Skew quadrupole:

The skew quadrupole potential in Cartesian coordinate can be written as:

$$\begin{aligned} \psi_2 &= \frac{A_2}{2 \cdot a} \cdot r^2 \sin\left[2 \cdot \left(\theta - \frac{\pi}{4}\right)\right] \\ &= \frac{-A_2}{2 \cdot a} \cdot r^2 \cos(2\theta) \\ &= \frac{-A_2}{2 \cdot a} \cdot r^2 \left(\cos^2\theta - \sin^2\theta\right) \\ &= \frac{-A_2}{2 \cdot a} \cdot \left(x^2 - y^2\right) \end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned} A_x &= \frac{A_2}{a} \cdot (-x) \\ A_y &= \frac{A_2}{a} \cdot y \end{aligned} \tag{3b}$$

Sextupole

The sextupole potential, n = 3, in Cartesian coordinate can be written as:

$$\begin{aligned} \varphi_3 &= \frac{B_3}{6 \cdot a^2} \cdot r^3 \sin(3 \cdot \theta) \\ &= \frac{B_3}{6 \cdot a^2} \cdot r^3 \cdot \left(3\cos^2\theta \cdot \sin\theta - \sin^3\theta\right) \\ &= \frac{B_3}{6 \cdot a^2}\left[3x^2 y - y^3\right] \end{aligned}$$

From this we derive the magnetic field:

$$\begin{aligned} B_x &= \frac{B_3}{a^2} \cdot [xy] \\ B_y &= \frac{B_3}{2 \cdot a^2} \cdot \left[x^2 - y^2\right] \end{aligned} \tag{4a}$$

Skew sextupole

The skew sextupole potential, n = 3, in Cartesian coordinate can be written as:

$$\psi_3 = \frac{A_3}{6\cdot a^2}\cdot r^3 \sin\left[3\cdot\left(\theta-\frac{\pi}{6}\right)\right]$$

$$= \frac{-A_3}{6\cdot a^2}\cdot r^3\cos(3\theta)$$

$$= \frac{-A_3}{6\cdot a^2}\cdot r^3\left(\cos^3\theta-3\cos\theta\cdot\sin^2\theta\right)$$

$$= \frac{-A_3}{6\cdot a^2}\left(x^3-3xy^2\right)$$

From this we derive the magnetic field:

$$A_x = \frac{-A_3}{2a^2}\cdot\left(x^2-y^2\right)$$
$$A_y = \frac{A_3}{a^2}\cdot(xy) \qquad (4b)$$

Octupole

The octupole potential, n = 4, in Cartesian coordinate can be written as:

$$\varphi_4 = \frac{B_4}{24\cdot a^3}\cdot r^4\sin(4\cdot\theta)$$

$$= \frac{B_4}{6\cdot a^3}\left[x^3y-xy^3\right]$$

From this we derive the magnetic field:

$$B_x = \frac{B_4}{6\cdot a^3}\cdot\left[3x^2y-y^3\right]$$
$$B_y = \frac{B_4}{6\cdot a^3}\cdot\left[x^3-3xy^2\right] \qquad (5a)$$

Skew octupole

The skew octupole potential, n = 4, in Cartesian coordinate can be written as:

$$\psi_4 = \frac{-A_4}{24\cdot a^3}\cdot r^4\cos(4\cdot\theta)$$

$$= \frac{-A_4}{24\cdot a^3}\cdot r^4\cdot\left(\cos^4\theta-6\sin^2\theta\cdot\cos^2\theta+\sin^4\theta\right)$$

$$= \frac{-A_4}{24\cdot a^3}\left[x^4-6x^2y^2+y^4\right]$$

From this we derive the magnetic field:

$$A_x = \frac{-A_4}{6 \cdot a^3} \cdot \left[x^3 - 3xy^2 \right]$$
$$A_y = \frac{A_4}{6 \cdot a^3} \cdot \left[3x^2 y - y^3 \right] \tag{5b}$$

Decapole

The decapole potential, n = 5, in Cartesian coordinate can be written as:

$$\begin{aligned} \varphi_5 &= \frac{B_5}{120 \cdot a^4} \cdot r^5 \sin(5 \cdot \theta) \\ &= \frac{B_5}{120 \cdot a^4} \cdot r^5 \cdot \left(5\cos^4\theta \cdot \sin\theta - 10\cos^2\theta \cdot \sin^3\theta + \sin^5\theta \right) \\ &= \frac{B_5}{120 \cdot a^4} \left[5x^4 y - 10x^2 y^3 + y^5 \right] \end{aligned}$$

From this we derive the magnetic field:

$$B_x = \frac{B_5}{6 \cdot a^4} \cdot \left[x^3 y - xy^3 \right]$$
$$B_y = \frac{B_5}{24 \cdot a^4} \cdot \left[x^4 - 6x^2 y^2 + y^4 \right] \tag{6a}$$

Skew decapole:

The skew decapole potential, n = 5, in Cartesian coordinate can be written as:

$$\begin{aligned} \psi_5 &= \frac{-A_5}{120 \cdot a^4} \cdot r^5 \cos(5 \cdot \theta) \\ &\frac{-A_5}{120 \cdot a^4} \cdot r^5 \cdot \left(\cos^5\theta - 10\cos^3\theta \cdot \sin^2\theta + 5\cos\theta \cdot \sin^4\theta \right) \\ &= \frac{-A_5}{120 \cdot a^4} \left[x^5 - 10x^3 y^2 + 5xy^4 \right] \end{aligned}$$

From this we derive the magnetic field:

$$A_x = \frac{-A_5}{24 \cdot a^4} \cdot \left[x^4 - 6x^2 y^2 + y^4 \right]$$
$$A_y = \frac{A_5}{6 \cdot a^4} \cdot \left[x^3 y - xy^3 \right] \tag{6b}$$

Rolled multipoles

Rolling the multipole by an angle +ε such that

$$\theta = \bar{\theta} + \varepsilon$$

and following equation (1) the vector potential of the *n*-th multipole can be written as:

$$\varphi_n = \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot \bar{r}^n \sin(n\bar{\theta}), \text{ with } \bar{r} = r.$$

In the original frame of reference this will be:

$$\begin{aligned}
\varphi_n &= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin\left[n\cdot(\theta-\varepsilon)\right] \\
&= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin(n\theta - n\varepsilon) \\
&= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \left[\sin(n\theta)\cos(n\varepsilon) - \cos(n\theta)\sin(n\varepsilon)\right] \qquad (7) \\
&= \frac{B_n \cos(n\varepsilon)}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin(n\theta) - \frac{B_n \sin(n\varepsilon)}{(n!)\cdot a^{(n-1)}} \cdot r^n \cos(n\theta) \\
&= N_n r^n \sin(n\theta) + S_n r^n \cos(n\theta)
\end{aligned}$$

where $N_n = \frac{B_n \cos(n\varepsilon)}{(n!)\cdot a^{(n-1)}}$ is the n-th order normal component and $S_n = \frac{-B_n \sin(n\varepsilon)}{(n!)\cdot a^{(n-1)}}$ is the *n*-th order skew component.

Given respective normal and skew components equation (6) can also be used in reverse to figure out the equivalent roll angle of a multipole field.

Flipped-around multipoles

Normal multipole reversal

When a multipole magnet is used in reverse, i.e. when beam is approaching in the opposite direction, the angular variable θ is replaced with $\pi - \theta$ in equation (1a). The vector potential of the n-th multipole can be written down as:

$$\begin{aligned}
\varphi_n &= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin\left\{n\cdot(\pi-\theta)\right\} \\
&= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin(n\cdot\pi - n\cdot\theta) \\
&= \frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \left[\sin(n\cdot\pi)\cdot\cos(n\cdot\theta) - \cos(n\cdot\pi)\cdot\sin(n\cdot\theta)\right] \\
&= -\cos(n\cdot\pi)\cdot\frac{B_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \sin(n\cdot\theta)
\end{aligned}$$

Therefore dipole, sextupole, and decapole, with n = *odd*, will not change sign. On the other hand quadrupole and octupole, with n = *even*, will change sign.

Skew multipole reversal

Replacing θ with π - θ in equation (1b), the vector potential of the n-th skew multipole can be written down as:

$$\begin{aligned}
\psi_n &= \frac{-A_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \cos\left[n\cdot(\pi-\theta)\right] \\
&= \frac{-A_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \cos(n\cdot\pi - n\cdot\theta) \\
&= \frac{-A_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \cdot\left[\cos(n\cdot\pi)\cos(n\cdot\theta) + \sin(n\cdot\pi)\sin(n\cdot\theta)\right] \\
&= \cos(n\cdot\pi)\cdot\frac{-A_n}{(n!)\cdot a^{(n-1)}} \cdot r^n \cos(n\cdot\theta)
\end{aligned}$$

Therefore, skew quadrupole and skew octupole, with n = *even*, will not change sign. Skew dipole, skew sextupole, and skew decapole, on the other hand, are with n = *odd* and will need to change sign.

About the only reason one would ever try to switch direction is to do calculation in both proton and anti-proton direction. Since anti-protons are with negative charge an overall sign change is expected, one way or the other.